Hedging discontinuous stochastic volatility models

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Abstract

We consider a stochastic volatility model with jumps where the underlying asset price is driven by a process sum of a 2-dimensional Brownian motion and 2-dimensional compensated Poisson process. The market is incomplete, there is an infinity of Equivalent Martingale Measures (E.M.M) and an infinity of hedging strategies. We characterize the set of E.M.M, and we hedge by minimizing the variance using Malliavin calculus.

Keywords: stochastic volatility model with jumps, incomplete markets, Malliavin calculus, Clark-Ocone formula, European options, equivalent martingale measure, mean-variance hedging.

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1 Introduction

Stochastic volatility models were introduced in the financial literature to take in account the *smile* effect. Most of works on these models assumes -for simplification- the continuity of the asset price trajectories (driven by Brownian motion). But an asset price can jump at any moment and randomly. We are interested here by discontinuous dynamic for the asset price with discontinuous stochastic volatility.

Formally, let the underlying asset price is given by

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(t, Y_t)[a_t^{(1)} dW_t^{(1)} + a_t^{(3)} dM_t^{(1)}], \quad t \in [0, T], \quad S_0 = x > 0,$$

with

$$dY_t = \mu_t^Y dt + \sum_{i=1}^{2} \sigma_t^{(i)} [a_t^{(i)} dW_t^{(i)} + a_t^{(i+2)} dM_t^{(i)}], \quad Y_0 = y \in \mathbb{R}.$$

$W = (W^{(1)}, W^{(2)})$ is a 2-dimensional Brownian motion, $M = (M^{(1)}, M^{(2)})$ is a 2-dimensional compensated Poisson process with independent components and deterministic intensity $(\int_0^t \lambda_s^{(1)} ds, \int_0^t \lambda_s^{(2)} ds)$ and for $1 \leq i \leq 4$, $a^{(i)} : [0, T] \longrightarrow \mathbb{R}$ is a deterministic function.

The most serious problem in a stochastic volatility model is the incompleteness. These models involve the existence of an infinity of equivalent martingale measures (EMM) i.e a probability

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equivalent to the historical one under which the actualized prices are martingales. First we seek a characterization for an EMM. We show that a probability Q equivalent to the historical probability P is specified by its Radon-Nikodym density w.r.t P

$$\begin{aligned}\rho_T &= \prod_{i=1}^{2} \exp\left(\int_0^T \beta_s^{(i)} dW_s^{(i)} - \frac{1}{2}\int_0^T (\beta_s^{(i)})^2 ds\right) \exp\left(\int_0^T \ln(1+\beta_s^{(i+2)}) dM_s^{(i)}\right.\\ &\quad \left. + \int_0^T \lambda_s^{(i)} \left[\ln(1+\beta_s^{(i+2)}) - \beta_s^{(i+2)}\right] ds\right),\end{aligned}$$

where $(\beta_t)_{t\in[0,T]}$ is a $\mathbb{R}^4$-valued predictable process such that $\beta^{(3)}, \beta^{(4)} > -1$. If Q is a $P-$EMM, $\beta^{(1)}$ and $\beta^{(3)}$ are related by

$$\mu_t - r_t + \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) + \lambda_t^{(1)} \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) = 0,$$

see Proposition 3.1.

The process $\left(-\frac{\mu_t - r_t}{a_t^{(1)}\sigma(t,Y_t)}, 0, 0, 0\right)$ is an example of $\mathbb{R}^4$-valued predictable process satisfying the above equation, and it defines a P-E.M.M. This means that the set of P-EMM is not empty. Moreover, since $\beta^{(2)}$ and $\beta^{(4)}$ doesn't appear in the last equation so they can be choosing arbitrarily and thus there exists an infinity of $P-$EMM.

Mean-variance hedging

In complete market, we have a unique hedging strategy. This is not the case for incomplete market. We have an infinity of hedging strategies. We hedge using the mean-variance hedging approach initiated by Föllmer and Sondermann (1986), and we find the strategy by applying Malliavin calculus.

Consider an option with payoff $f(S_T)$ where $(S_t)_{t\in[0,T]}$ is the asset price and with maturity T. We work with a P-E.M.M $\hat{Q}$. Let $(\hat{\eta}_t, \hat{\zeta}_t)_{t\in[0,T]}$ be a self-hedging strategy and $(\hat{V}_t)_{t\in[0,T]}$ be the portfolio value process. We get using the chaotic calculus, that the strategy minimizing the variance $E_{\hat{Q}}\left[(f(S_T) - \hat{V}_T)^2\right]$, is given by

$$\hat{\eta}_t = \frac{a_t^{(1)} E[D_t^{\hat{W}^{(1)}} f(S_T) \mid \mathcal{F}_t] + \lambda_t^{(1)}(1+\hat{\beta}_t^{(3)}) a_t^{(3)} E[D_t^{N^{(1)}} f(S_T) \mid \mathcal{F}_t]}{((a_t^{(1)})^2 + \lambda_t^{(1)}(1+\hat{\beta}_t^{(3)})(a_t^{(3)})^2) e^{\int_t^T r_s ds} \sigma(t, Y_t) S_t},$$

where $\hat{W}_t^{(1)} = W_t^{(1)} - \int_0^t \hat{\beta}^{(1)} ds$, and the operators $D^{\hat{W}^{(1)}}$ and $D^{N^{(1)}}$ are respectively the Malliavin derivative in the direction of the one dimensional Brownian motion $\hat{W}^{(1)}$ and the Malliavin operator in the direction of the Poisson process $N^{(1)}$.

The paper is organized as follows : In Section 2, we present some necessary formulas. In the

third section we introduce the model. The fourth one is devoted to the hedging by minimizing the variance via Malliavin calculus. In the last section, we characterize the E.M.M minimizing the entropy, this allows us to establish explicit formulae for the strategy.

2 Preliminary

Let $W = (W^{(1)}, W^{(2)})$ be a 2-dimensional Brownian motion and $N = (N^{(1)}, N^{(2)})$ denotes a 2-dimensional Poisson process with independent components and deterministic intensity $(\int_0^t \lambda_s^{(1)} ds, \int_0^t \lambda_s^{(2)} ds)$. We work in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the naturel filtration generated by W and N. We denote by $M = (M^{(1)}, M^{(2)})$ the associated compensated Poisson process i.e for $i = 1, 2$ and $t \in [0, T]$ we have $dM_t^{(i)} = dN_t^{(i)} - \lambda_t^{(i)} dt$. Both $(\mathcal{F}_t)_{t\in[0,T]}$-martingales W and M are independent.

Definition 2.1 *Let Γ be the set of all $\mathcal{F}_t$-predictable processes $(\gamma_t)_{t\in[0,T]}$ with values in $\mathbb{R}^4$, such that*

$$\sum_{i=1}^{2} E_P\left[\int_0^t (\gamma_s^{(i)})^2 ds\right] + \sum_{i=1}^{2} E_P\left[\int_0^t (\gamma_s^{(i+2)})^2 \lambda_s^i ds\right] < \infty, \quad t \in [0, T].$$

We denote by $\mathcal{E}(X)_t$, for a semi-martingale X with $X_0 = 0$, the unique solution of the stochastic differential equation

$$Z_t = 1 + \int_0^t Z_{s^-} dX_s.$$

$\mathcal{E}(X)_t$ is called the Doléans-Dade exponential. We have (Theorem 36 of Protter (1990))

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X_t, X_t]^c\right) \prod_{s\le t}(1 + \Delta X_s)\exp(-\Delta X_s). \tag{2.1}$$

Remark 2.1 *Notice that for $\gamma \in \Gamma$ such that $\gamma^{(3)}, \gamma^{(4)} > -1$ we have for $i = 1, 2$*

$$\begin{aligned}
\mathcal{E}(\gamma^{(i)} W^{(i)})_t &= \exp\left(\int_0^t \gamma_s^{(i)} dW_s^{(i)} - \frac{1}{2}\int_0^t (\gamma_s^{(i)})^2 ds\right), \\
\mathcal{E}(\gamma^{(i+2)} M^{(i)})_t &= \exp\left(\int_0^t \ln(1 + \gamma_s^{(i+2)}) dM_s^{(i)}\right. \\
&\quad \left. + \int_0^t \lambda_s^{(i)} \left[\ln(1 + \gamma_s^{(i+2)}) - \gamma_s^{(i+2)}\right] ds\right).
\end{aligned}$$

The next lemma is the martingale representation theorem (Jacod (1979)).

Lemma 2.1 *Let $Z = (Z_t)_{t\in[0,T]}$ be a $\mathcal{F}_t$-martingale. There exists a predictable process $\gamma \in \Gamma$ such that*

$$dZ_t = \sum_{i=1}^{2} \gamma_t^{(i)} dW_t^{(i)} + \sum_{i=1}^{2} \gamma_t^{(i+2)} dM_t^{(i)}, \quad t \in [0, T].$$

The Itô formula is given by the following lemma (see Protter (1990)).

Lemma 2.2 *Let $R = (R^1, \ldots, R^n)$ be a n-dimensional adapted process, and $\gamma = (\gamma^1, \ldots, \gamma^n)$ such that*

$$\gamma^k = (\gamma^{(k,1)}, \gamma^{(k,2)}, \gamma^{(k,3)}, \gamma^{(k,4)}) \in \Gamma, \quad 1 \leq k \leq n.$$

We consider the process $X = (X^1, \ldots, X^n)$ where for $k \in \{1, \ldots, n\}$, X^k is given by

$$dX_t^k = R_t^k dt + \sum_{i=1}^{2} \gamma_t^{(k,i)} dW_t^{(i)} + \sum_{i=1}^{2} \gamma_t^{(k,i+2)} dM_t^{(i)}, \quad X_0^k \in \mathbb{R}.$$

For any function $f \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$, we have

$$\begin{aligned}
f(t, X_t) \quad = \quad & f(0, X_0) + \int_0^t \left[\frac{\partial f}{\partial s}(s, X_{s^-}) + \langle R_s, \nabla f(s, X_s) \rangle \right. \\
& + \sum_{k,l=1}^{n} \frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2 f}{\partial x^k x^l}(s, X_{s^-}) \gamma_s^{(k,i)} \gamma_s^{(l,i)} \\
& + \sum_{k=1}^{n} \sum_{i=1}^{2} \lambda_s^{(i)} \Big(f(s, (X_{s^-}^1, \ldots, X_{s^-}^k + \gamma_s^{(k,i+2)}, \ldots, X_{s^-}^n)) - f(s, X_{s^-}) \\
& \left. - \gamma_s^{(k,i+2)} \frac{\partial f}{\partial x^k}(s, X_{s^-}) \Big) \right] ds + \sum_{k=1}^{n} \sum_{i=1}^{2} \int_0^t \gamma_s^{(k,i)} \frac{\partial f}{\partial x^k}(s, X_s) dW_s^{(i)} \\
& + \sum_{k=1}^{n} \sum_{i=1}^{2} \int_0^t (f(s, (X_{s^-}^1, \ldots, X_{s^-}^k + \gamma_s^{(k,i+2)}, \ldots, X_{s^-}^n)) - f(s, X_{s^-})) dM_s^{(i)}.
\end{aligned}$$

3 The model

Let us consider a market with two assets: a risky asset to which is related a European call option and a riskless one. The maturity is T and the strike is K. The price of the riskless asset is given by

$$dA_t = r_t A_t dt, \quad t \in [0,T], \quad A_0 = 1,$$

where r_t is deterministic and denotes the interest rate. The price of the risky asset has a stochastic volatility and is given by

$$\begin{aligned}
\frac{dS_t}{S_t} \quad &= \quad \mu_t dt + \sigma(t, Y_t)[a_t^{(1)} dW_t^{(1)} + a_t^{(3)} dM_t^{(1)}], \quad t \in [0,T], \quad S_0 = x > 0, \qquad (3.1) \\
dY_t \quad &= \quad \mu_t^Y dt + \sum_{i=1}^{2} \sigma_t^{(i)} [a_t^{(i)} dW_t^{(i)} + a_t^{(i+2)} dM_t^{(i)}], \quad t \in [0,T], \quad Y_0 = y \in \mathbb{R},
\end{aligned}$$

where for $1 \leq i \leq 4$, $a^{(i)} : [0,T] \longrightarrow \mathbb{R}$ is a deterministic function. We assume that

$$\sigma(t,.) \neq 0, \quad \text{and} \quad 1 + \sigma(t,.) a_t^{(3)} > 0, \quad t \in [0,T].$$

We have

$$
\begin{aligned}
S_t &= x\exp\left(\int_0^t a_s^{(1)}\sigma(s,Y_s)dW_s^{(1)}+\int_0^t(\mu_s-a_s^{(3)}\lambda_s^{(1)}\sigma(s,Y_s)-\frac{1}{2}(a_s^{(1)})^2\sigma^2(s,Y_s))ds\right)\\
&\quad\times\prod_{k=1}^{k=N_t}(1+a^{(3)}_{T_k^{(1)}}\sigma(T_k^{(1)},Y_{T_k^{(1)}})),
\end{aligned}
$$

$0\le t\le T$, where $(T_k^{(1)})_{k\ge 1}$ denotes the jump times of $(N_t^{(1)})_{t\in[0,T]}$.

3.1 Change of probability

Let Q be a P-equivalent probability; by the Radon-Nikodym theorem there exists a $\mathcal{F}_T$-measurable random variable, $\rho_T:=\frac{dQ}{dP}$, such that $Q(A)=E_P[\rho_T 1_A]$, $A\in\mathcal{P}(\Omega)$. Notice that ρ_T is strictly positive P-a.s, since Q is equivalent to P, and $E_P[\rho_T]=E_P[\rho_T 1_\Omega]=1$. Consider now the P-martingale $\rho=(\rho_t)_{t\in[0,T]}$ defined by

$$\rho_t:=E_P[\rho_T\mid\mathcal{F}_t]=E_P\left[\frac{dQ}{dP}\mid\mathcal{F}_t\right].$$

Definition 3.1 *Let $\mathcal{H}$ be the set of all $P-EMM$ i.e $Q\in\mathcal{H}$ if and only if $Q\simeq P$ and the actualized prices are Q-martingales.*

The next proposition gives the Radon-Nikodym density w.r.t P of a P-EMM.

Proposition 3.1 *Let $Q\in\mathcal{H}$. There exists a predictable process $(\beta_t)_{t\in[0,T]}$ taking values in $\mathbb{R}^4$ such that $\beta^{(3)},\beta^{(4)}>-1$ and the Radon-Nikodym density of Q w.r.t P is given by*

$$
\begin{aligned}
\rho_T &= \prod_{i=1}^{2}\mathcal{E}(\beta^{(i)}W^{(i)})_T\mathcal{E}(\beta^{(i+2)}M^{(i)})_T\\
&= \prod_{i=1}^{2}\exp\left(\int_0^T\beta_s^{(i)}dW_s^{(i)}-\frac{1}{2}\int_0^T(\beta_s^{(i)})^2ds\right)\exp\left(\int_0^T\ln(1+\beta_s^{(i+2)})dM_s^{(i)}\right.\\
&\quad\left.+\int_0^T\lambda_s^{(i)}\left[\ln(1+\beta_s^{(i+2)})-\beta_s^{(i+2)}\right]ds\right). \qquad (3.2)
\end{aligned}
$$

Moreover $\beta^{(1)}$ and $\beta^{(3)}$ are related by

$$\mu_t-r_t+\beta_t^{(1)}a_t^{(1)}\sigma(t,Y_t)+\lambda_t^{(1)}\beta_t^{(3)}a_t^{(3)}\sigma(t,Y_t)=0. \qquad (3.3)$$

Proof. We follow Bellamy (1999) for the case of discontinuous market with deterministic volatility. By the martingale representation theorem (Lemma 2.1) there exists a predictable process $(\gamma_t)_{t\in[0,T]}\in\Gamma$ such that

$$d\rho_t=\sum_{i=1}^{2}\gamma_t^{(i)}dW_t^{(i)}+\sum_{i=1}^{2}\gamma_t^{(i+2)}dM_t^{(i)},\quad t\in[0,T].$$

We have $P(\rho_t > 0, t \in [0,T]) = 1$; assuming $\beta := \frac{\gamma}{\rho}$, we obtain

$$\frac{d\rho_t}{\rho_t} = \sum_{i=1}^{2} \beta_t^{(i)} dW_t^{(i)} + \sum_{i=1}^{2} \beta_t^{(i+2)} dM_t^{(i)} = dX_t, \quad t \in [0,T].$$

(3.2) follows from (2.1). In addition $(e^{-\int_0^t r_s ds} S_t)_{t\in[0,T]}$ is a Q-martingale, which is equivalent to say that $(e^{-\int_0^t r_s ds} S_t \rho_t)_{t\in[0,T]}$ is a P-martingale. The integration by parts formula (Protter (1990)) gives

$$d(e^{-\int_0^t r_s ds} S_t \rho_t) = \rho_t d(e^{-\int_0^t r_s ds} S_t) + e^{-\int_0^t r_s ds} S_t d\rho_t + d[e^{-\int_0^t r_s ds} S_t, \rho_t],$$

with

$$\begin{aligned} d[e^{-\int_0^t r_s ds} S_t, \rho_t] &= \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) dt + \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) dN_t^{(1)}, \\ &= \left(\beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) + \lambda_t^{(1)} \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t)\right) dt + \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) dM_t^{(1)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} d(e^{-\int_0^t r_s ds} S_t \rho_t) &= \rho_t S_t e^{-\int_0^t r_s ds} \Big[(\mu_t - r_t + \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) + \lambda_t^{(1)} \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t)) dt \\ &\quad + (\beta_t^{(1)} + \sigma(t, Y_t) a_t^{(1)}) dW_t^{(1)} + \beta_t^{(2)} dW_t^{(2)} \\ &\quad + \left(\sigma(t, Y_t) a_t^{(3)} + \beta_t^{(3)} (1 + \sigma(t, Y_t) a_t^{(3)})\right) dM_t^{(1)} + \beta_t^{(4)} dM_t^{(2)} \Big]. \end{aligned}$$

Thus Q is a P-EMM if

$$\mu_t - r_t + \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) + \lambda_t^{(1)} \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) = 0.$$

$\square$

One can notice that there is no restriction on $\beta^{(2)}$ and $\beta^{(4)}$, which means that if $\mathcal{H} \neq \emptyset$ thus $\mathcal{H}$ contains an infinity of P-EMM.

3.2 Girsanov theorem

Let $\Gamma^{\mathcal{H}}$ be the set of processes $\beta \in \Gamma$ satisfying (3.3). The Radon-Nikodym derivative ρ_T associated to β and given by (3.2) define a P-EMM. From now on, a P-EMM Q in $\mathcal{H}$ will be denoted by Q^β where $\beta \in \Gamma^{\mathcal{H}}$.

Let $\beta \in \Gamma^{\mathcal{H}}$ and consider the two processes $\tilde{W} = (\tilde{W}^{(1)}, \tilde{W}^{(2)})$ and $\tilde{M} = (\tilde{M}^{(1)}, \tilde{M}^{(2)})$ where for $i = 1, 2$

$$\tilde{W}_t^{(i)} = W_t^{(i)} - \int_0^t \beta^{(i)} ds, \quad t \in [0,T], \quad \text{and} \quad \tilde{M}_t^{(i)} = M_t^{(i)} - \int_0^t \lambda_s^{(i)} \beta_s^{(i+2)} ds, \quad t \in [0,T].$$

By Girsanov theorem (Jacod (1979)) $\tilde{W}$ is a Q^β-Brownian motion and $\tilde{M}$ is a Q^β-compensated Poisson process. The dynamic of $(S_t)_{t\in[0,T]}$ under Q^β is

$$\frac{dS_t}{S_t} = r_t dt + \sigma(t,Y_t)[a_t^{(1)} d\tilde{W}_t^{(1)} + a_t^{(3)} d\tilde{M}_t^{(1)}], \quad t\in[0,T], \quad S_0 = x > 0,$$

and $(Y_t)_{t\in[0,T]}$ is given by

$$\begin{aligned} dY_t &= \left(\mu_t^Y + \sum_{i=1}^2 \sigma_t^{(i)}[a_t^{(i)}\beta_t^{(i)} + \lambda_t^{(i)}\beta_t^{(i+2)}a_t^{(i+2)}]\right) dt \\ &\quad + \sum_{i=1}^2 \sigma_t^{(i)}[a_t^{(i)} d\tilde{W}_t^{(i)} + a_t^{(i+2)} d\tilde{M}_t^{(i)}], \quad t\in[0,T] \quad Y_0 = y \in \mathbb{R}. \end{aligned}$$

4 Hedging by minimizing the variance

In this section we are interested by finding an optimal strategy for our model described in Section 3. We compute the strategy by minimizing the variance. This is on applying Malliavin calculus.

From now on, we work with the P-EMM minimizing the entropy $\hat{Q}$. The price of a European option with payoff $f(S_T)$ is $E_{\hat{Q}}\left[e^{-\int_t^T r_s ds} f(S_T) \mid \mathcal{F}_t\right], \quad t\in[0,T]$. Our aim is to determine the $\mathcal{F}_t$-adapted strategy $(\hat{\zeta}_t, \hat{\eta}_t)_{t\in[0,T]}$ that minimizes

$$E_{\hat{Q}}\left[(f(S_T) - \hat{V}_T)^2\right], \tag{4.1}$$

where $\hat{\zeta}_t$, $\hat{\eta}_t$ and $\hat{V}_t$ denote respectively the number of units invested in riskless and risky asset and the value of the portfolio. We have for $t\in[0,T]$ $\hat{V}_t = \hat{\zeta}_t A_t + \hat{\eta}_t S_t$. Since the strategy is assumed to be self-financing, so $dV_t = \hat{\zeta}_t dA_t + \hat{\eta}_t dS_t$ and

$$dV_t = r_t V_t dt + \sigma(t,Y_t)\hat{\eta}_t S_t [a_t^{(1)} d\hat{W}_t^{(1)} + a_t^{(3)} d\hat{M}_t^{(1)}], \quad t\in[0,T].$$

Therefore

$$\hat{V}_T = \hat{V}_0 e^{\int_0^T r_s ds} + \int_0^T e^{\int_t^T r_s ds}\sigma(t,Y_t)\hat{\eta}_t S_t [a_t^{(1)} d\hat{W}_t^{(1)} + a_t^{(3)} d\hat{M}_t^{(1)}]. \tag{4.2}$$

4.1 Chaotic calculus

The chaotic calculus allows to obtain the strategy that minimizes the variance, that is by using the Clark-Ocone formula.

Let us denote by $\hat{X}$, the 4-dimensional martingale coming from the 2-dimensional Brownian motion and compensated Poisson process introduced in section 2, i.e

$$(\hat{X}_t^{(1)}, \hat{X}_t^{(2)}, \hat{X}_t^{(3)}, \hat{X}_t^{(4)}) = (\hat{W}_t^{(1)}, \hat{W}_t^{(2)}, \hat{M}_t^{(1)}, \hat{M}_t^{(2)}), \quad t\in[0,T].$$

This martingale has the Chaotic Representation Property (CRP). The CRP for $\hat{X}$ states that any square-integrable functional $\mathcal{F}_T$-measurable, can be expanded into a series of multiple stochastic integrals -w.r.t $\hat{X}_t$- of deterministic functions. Using this expansion, we define the Malliavin operator, by acting on the multiple stochastic integrals. The Clark-Ocone formula is then deduced by technical ways, allowing to make appear the Malliavin operator in the expansion, and to write this last one as a simple stochastic integral w.r.t $\hat{X}$.

We define the multiple stochastic integral and introduce the Malliavin operator and the Clark-Ocone formula in the multidimensional Brownian-Poisson case (more precisely in the 4-dimensional case, the following definitions and formulas can be extended for the $d-$dimensional case, $d > 4$). For more details we refer to Løkka (1999), Nualart (1995), Nualart and Vives(1990), Øksendal (1996) and Privault (1997 a,b). Let (e_1, e_2, e_3, e_4) be the canonical base of $\mathbb{R}^4$. For $g_n \in L^2([0,T]^n)$ we define the n-th iterated stochastic integral of the function $f_n e_{i_1} \otimes \ldots \otimes e_{i_n}$, with $1 \leq i_1, \ldots, i_n \leq 4$, by

$$I_n(g_n e_{i_1} \otimes \ldots \otimes e_{i_n}) : \quad = \quad n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g_n(t_1, \ldots, t_n) d\hat{X}_{t_1}^{(i_1)} \ldots d\hat{X}_{t_n}^{(i_n)}.$$

The iterated stochastic integral of a symmetric function $f_n = (f_n^{(i_1,\ldots,i_n)})_{1 \leq i_1, \ldots, i_n \leq 4} \in$ $L^2([0,T], \mathbb{R}^4)^{\otimes n}$, where $f_n^{(i_1,\ldots,i_n)} \in L^2([0,T]^n)$, is

$$\begin{aligned} I_n(f_n) : \quad &= \sum_{i_1,\ldots,i_n=1}^{4} I_n(f_n^{(i_1,\ldots,i_n)} e_{i_1} \otimes \ldots \otimes e_{i_n}) \\ &= n! \sum_{i_1,\ldots,i_n=1}^{4} \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n^{(i_1,\ldots,i_n)}(t_1, \ldots, t_n) d\hat{X}_{t_1}^{(i_1)} \ldots d\hat{X}_{t_n}^{(i_n)}. \end{aligned}$$

For $F \in L^2(\Omega)$, there exists a unique sequence $(f_n)_{n \in \mathbb{N}}$ of deterministic symmetric functions $f_n = (f_n^{(i_1,\ldots,i_n)})_{i_1,\ldots,i_n \in \{1,\ldots,4\}} \in L^2([0,T], \mathbb{R}^4)^{\circ n}$ such that

$$F = \sum_{n=0}^{\infty} I_n(f_n). \tag{4.3}$$

Definition 4.1 *Let $l \in \{1, \ldots, 4\}$, we define the operator $D^{(l)} : \mathrm{Dom}\ (D^{(l)}) \subset L^2(\Omega) \to L^2(\Omega, [0,T])$ does correspond for $F \in \mathrm{Dom}\ (D^{(l)})$ (F having the representation (4.3)), the process $(D_t^{(l)} F)_{t \in [0,T]}$ given by*

$$\begin{aligned} D_t^{(l)} F := \sum_{n=1}^{\infty} \sum_{h=1}^{n} \sum_{i_1,\ldots,i_n=1}^{4} 1_{\{i_h = l\}} \\ I_{n-1}(f_n^{(i_1,\ldots,i_n)}(t_1, \ldots, t_{l-1}, t, t_{l+1} \ldots, t_n) e_{i_1} \otimes \ldots \otimes e_{i_{h-1}} \otimes e_{i_{h+1}} \ldots \otimes e_{i_n}) \end{aligned}$$

$$= \sum_{n=1}^{\infty} nI_{n-1}(f_n^l(*,t)), \quad d\hat{Q} \times dt - a.e.$$

with $f_n^l = (f_n^{(i_1,\ldots,i_{n-1},l)} e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})_{1 \le i_1,\ldots,i_{n-1} \le 4}$.

The domain of $D^{(l)}$ is

$$\begin{aligned} \text{Dom } (D^{(l)}) &= \Bigg\{ F = \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n=1}^{4} I_n(f_n^{(i_1,\ldots,i_n)} e_{i_1} \otimes \ldots \otimes e_{i_n}) \in L^2(\Omega) : \\ & \sum_{i_1,\ldots,i_n=1}^{4} \sum_{n=0}^{\infty} nn! \| f_n^{(i_1,\ldots,i_n)} \|^2_{L^2([0,T]^n)} < \infty \Bigg\}. \end{aligned}$$

We will now give the probabilistic interpretations of $D^{(l)}$ in the Brownian motion and Poisson process cases. These interpretations will allow us to compute the requested strategy.

The Brownian operator For $1 \le l \le 2$, the operator $D^{(l)}$ is, in fact, the Malliavin derivative in the direction of the one dimensional Brownian motion $\hat{W}^{(l)}$. So, we have for $1 \le l \le 2$ and $F = f\left(\hat{W}_{t_1}, \ldots, \hat{W}_{t_n}\right) \in L^2(\Omega)$ where $(t_1, \ldots, t_n) \in [0,T]^n$ and $f(x^{11}, x^{21}, \ldots, x^{1n}, x^{2n}) \in \mathcal{C}_b^\infty(\mathbb{R}^{2n})$

$$D_t^{(l)} F = \sum_{k=1}^{k=n} \frac{\partial f}{\partial x^{lk}} \left(\hat{W}_{t_1}, \ldots, \hat{W}_{t_n}\right) 1_{[0,t_k]}(t).$$

To calculate the Mallaivin derivative for Itô integral, we will use the following proposition (see corollary 5.13 of Øksendal (1996).

Proposition 4.1 *Let* $(u_t)_{t \in [0,T]}$ *be a* $\mathcal{F}_t-$*adapted process, such that* $u_t \in \text{Dom } (D^{(l)})$*, we have*

$$D_t^{(l)} \int_0^T u_s d\hat{W}_s^{(l)} = \int_t^T (D_t^{(l)} u_s) d\hat{W}_s^{(l)} + u_t,$$

The Poisson operator For $3 \le l \le 4$, $D^{(l)}$ is the Malliavin operator[†] in the direction of the Poisson process $N^{(l-2)}$. For $F \in \text{Dom } (D^{(l)})$

$$D_t^{(l)} F(\omega^{(1)}, \ldots, \omega^4) = \begin{cases} F(\omega^{(1)}, \omega^{(2)}, \omega^{(3)} + 1_{[t,\infty[}, \omega^{(4)}) - F(\omega^{(1)}, \ldots, \omega^{(4)}), & l = 3, \\ F(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \omega^{(4)} + 1_{[t,\infty[}) - F(\omega^{(1)}, \ldots, \omega^{(4)}), & l = 4. \end{cases}$$

The Clark-Ocone formula is given by the next proposition.

Proposition 4.2 *Consider a square-integrable functional* F, $\mathcal{F}_T$*-measurable, such that* $F \in \bigcap_{l=1}^4 \text{Dom } (D^{(l)})$. F *has the following predictable representation*

$$F = E[F] + \sum_{l=1}^{2} \int_0^T E[D_t^{(l)} F \mid \mathcal{F}_t] d\hat{W}_t^{(l)} + \sum_{l=1}^{2} \int_0^T E[D_t^{N^{(l)}} F \mid \mathcal{F}_t] d\hat{M}_t^{(l)}.$$

[†] Notice that, unlike the Brownian case, the Malliavin operator in the Poisson space does not a derivative.

Now we apply the Clark-Ocone formula to determine the strategy minimizing the variance for our model considered in the Section 3.

Proposition 4.3 *The strategy minimizing (4.1) of the model of Section 3 is given by*

$$\hat{\eta}_t = \frac{a_t^{(1)} E[D_t^{\hat{W}^{(1)}} f(S_T) \mid \mathcal{F}_t] + \lambda_t^{(1)}(1+\hat{\beta}_t^{(3)}) a_t^{(3)} E[D_t^{N^{(1)}} f(S_T) \mid \mathcal{F}_t]}{((a_t^{(1)})^2 + \lambda_t^{(1)}(1+\hat{\beta}_t^{(3)})(a_t^{(3)})^2) e^{\int_t^T r_s ds} \sigma(t, Y_t) S_t}. \tag{4.4}$$

Proof. First we approach the function $x \mapsto f(x)(= (x-K)^+ \text{or} = (K-x)^+)$ by polynomials on compact intervals and proceed as in Øksendal (1996)pp. 5-13. By dominated convergence, $(f(S_T) \in \bigcap_{l=1}^4 \text{Dom } (D^{(l)})$. Applying the Clark-Ocone formula to $f(S_T)$ and using (4.2) we obtain

$$\begin{aligned}
& E_{\hat{Q}}\left[f(S_T) - \hat{V}_T)^2\right] = \\
& \quad E_{\hat{Q}}\left[\left(\int_0^T \left(E[D_t^{\hat{W}^{(1)}} f(S_T) \mid \mathcal{F}_t] - e^{\int_t^T r_s ds} \sigma(t, Y_t) \hat{\eta}_t S_t a_t^{(1)}\right) d\hat{W}_t^{(1)}\right)^2\right. \\
& \quad + \left(\int_0^T E_{\hat{Q}}[D_t^{\hat{W}^{(2)}} f(S_T) \mid \mathcal{F}_t] d\hat{W}_t^{(2)}\right)^2 + \left(\int_0^T E_{\hat{Q}}[D_t^{N^{(2)}} f(S_T) \mid \mathcal{F}_t] d\hat{M}_t^{(2)}\right)^2 \\
& \quad \left. + \left(\int_0^T \left(E_{\hat{Q}}[D_t^{N^{(2)}} f(S_T) \mid \mathcal{F}_t] - e^{\int_t^T r_s ds} \sigma(t, Y_t) \hat{\eta}_t S_t a_t^{(3)}\right) d\hat{M}^1(t)\right)^2\right] \\
& = \quad E_{\hat{Q}}\left[\int_0^T h_2(\hat{\eta}_t) dt\right],
\end{aligned}$$

where

$$\begin{aligned}
h_2(x) \quad = \quad & (E_{\hat{Q}}[D_t^{\hat{W}^{(2)}} f(S_T) \mid \mathcal{F}_t])^2 + \lambda_t^{(2)}(1+\hat{\beta}_t^{(4)})(E[D_t^{N^{(2)}} f(S_T) \mid \mathcal{F}_t])^2 \\
& + \left(E_{\hat{Q}}[D_t^{\hat{W}^{(1)}} f(S_T) \mid \mathcal{F}_t] - e^{\int_t^T r_s ds} \sigma(t, Y_t) x S_t a_t^{(1)}\right)^2 \\
& + \lambda_t^{(1)}(1+\hat{\beta}_t^{(3)}) \left(E_{\hat{Q}}[D_t^{N^{(1)}} f(S_T) \mid \mathcal{F}_t] - e^{\int_t^T r_s ds} \sigma(t, Y_t) x S_t a_t^{(3)}\right)^2.
\end{aligned}$$

It is easily verified that h_2 is convex, hence the minimum is the solution of $h_2^{'}(x) = 0$. Therefore the strategy minimizing the variance is given by (4.4). □

5 Explicit formulae, Equivalent Martingale Measure minimizing the entropy

The process $\left(\frac{\mu_t - r_t}{a_t^{(1)} \sigma(t, Y_t)}, 0, 0, 0\right)$ belongs to $\Gamma^{\mathcal{H}}$ and it defines a P-EMM, so $\mathcal{H} \neq \emptyset$. Thus $\mathcal{H}$ contains an infinity of P-EMM. We choose the one that minimizes the relative entropy. Let $Q^\beta \in \mathcal{H}$. Denoting by $I(Q^\beta, P)$ the relative entropy of Q^β w.r.t P, we have

$$I(Q^\beta, P) = E_P\left[\frac{dQ^\beta}{dP} \ln \frac{dQ^\beta}{dP}\right].$$

Our aim is to minimize $I(P,Q^\beta)$ under $\mathcal{H}$. We have

$$I(P,Q^\beta)=E_{Q^\beta}\left[\frac{dP}{dQ^\beta}\ln\frac{dP}{dQ^\beta}\right]$$

Therefore the problem is to find a $\hat{\beta}$ which satisfy

$$I(P,Q^{\hat{\beta}})=\min_{\beta\in\Gamma^{\mathcal{H}}}-E_P\left[\ln\frac{dQ^\beta}{dP}\right]. \tag{5.1}$$

Lemma 5.1 *The minimization problem (5.1) is equivalent to the minimization of*

$$\begin{aligned}&(\mu_t-r_t+\lambda_t^{(1)}a_t^{(3)}\beta_t^{(3)}\sigma(t,Y_t))^2-2\sigma^2(t,Y_t)(a_t^{(1)})^2\lambda_t^{(1)}\left[\ln(1+\beta_t^{(3)})-\beta_t^{(3)}\right]\\&-2\sigma^2(t,Y_t)(a_t^{(1)})^2\lambda_t^{(2)}\left[\ln(1+\beta_t^{(4)})-\beta_t^{(4)}\right],\end{aligned}$$

under all $\beta=\left(\frac{\mu_t-r_t+\lambda_t^{(1)}a_t^{(3)}\sigma(t,Y_t)\beta_t^{(3)}}{\sigma(t,Y_t)a_t^{(1)}},0,\beta^{(3)},\beta^{(4)}\right)\in\Gamma^{\mathcal{H}}$.

Proof. Let $Q^\beta\in\mathcal{H}$. By (3.2)

$$\begin{aligned}I(P,Q^\beta)&=-E_P\left[\ln\frac{dQ^\beta}{dP}\right]\\&=E_P\left[\int_0^T\sum_{i=1}^2\frac{1}{2}(\beta_t^{(i)})^2-\lambda_t^{(i)}\left[\ln(1+\beta_t^{(i+2)})-\beta_t^{(i+2)}\right]dt\right]\\&=E_P\left[\int_0^T\frac{G(\beta_t)}{2\sigma^2(t,Y_t)(a_t^{(1)})^2}dt\right],\end{aligned}$$

where $\beta\in\Gamma^{\mathcal{H}}$, and G is the function defined by

$$\begin{aligned}G(\beta_t)&=2\sigma^2(t,Y_t)(a_t^{(1)})^2\left(\frac{1}{2}(\beta_t^{(1)})^2+\frac{1}{2}(\beta_t^{(2)})^2-\lambda_t^{(1)}\left[\ln(1+\beta_t^{(3)})-\beta_t^{(3)}\right]\right.\\&\qquad\left.-\lambda_t^{(2)}\left[\ln(1+\beta_t^{(4)})-\beta_t^{(4)}\right]\right),\quad t\in[0,T].\end{aligned}$$

For a fixed t, we have by (3.3),

$$\begin{aligned}G(\beta_t)&=(\mu_t-r_t+\lambda_t^{(1)}a_t^{(3)}\sigma(t,Y_t)\beta_t^{(3)})^2+\sigma^2(t,Y_t)(a_t^{(1)})^2\left((\beta_t^{(2)})^2\right.\\&\qquad\left.-2\lambda_t^{(1)}\left[\ln(1+\beta_t^{(3)})-\beta_t^{(3)}\right]-2\lambda_t^{(2)}\left[\ln(1+\beta_t^{(4)})-\beta_t^{(4)}\right]\right),\quad t\in[0,T].\end{aligned}$$

Now the lemma is deduced from the fact that $\beta_t^{(2)}$ appears only in the term $\sigma^2(t,Y_t)(a_t^{(1)})^2(\beta_t^{(2)})^2$ which is always positive : so $\beta_t^{(2)}$ must be equal to zero. $\square$

The following proposition gives the solution to the minimization 5.1.

Proposition 5.1 *Consider* $(\hat{\beta}_t^{(1)}, \hat{\beta}_t^{(2)}, \hat{\beta}_t^{(3)}, \hat{\beta}_t^{(4)})_{t\in[0,T]} \in \Gamma^{\mathcal{H}}$*, with*

$$\hat{\beta}_t^{(2)} = \hat{\beta}_t^{(4)} = 0, \quad \hat{\beta}_t^{(1)} = \begin{cases} \frac{r_t - \mu_t - \lambda_t^{(1)} a_t^{(3)} \sigma(t,Y_t)\hat{\beta}_t^{(3)}}{\sigma(t,Y_t) a_t^{(1)}} & \text{if} \quad a^{(1)} \neq 0, \\ 0 & \text{if} \quad a^{(1)} = 0, \end{cases}$$

and let $\hat{\beta}_t^{(3)}$ *be the unique solution of the equation*

$$\lambda_t^{(1)} \sigma(t,Y_t)(a_t^{(3)})^2 x + (a_t^{(1)})^2 \sigma(t,Y_t)\left(\frac{x}{1+x}\right) - a_t^{(3)}(r_t - \mu_t) = 0. \tag{5.2}$$

Then, the P-EMM $\hat{Q}$ *defined by its Radon-Nikodym density*

$$\prod_{i=1}^{2} \mathcal{E}(\hat{\beta}^{(i)} W^{(i)})_T \mathcal{E}(\hat{\beta}^{(i+2)} M^{(i)})_T,$$

is the P-EMM minimizing $I(P, Q^\beta)$*.*

Proof. By Lemma 5.1, we have to minimize the function $F :]-1,\infty[\times]-1,\infty[\longrightarrow \mathbb{R}$ defined by

$$F(x,y) \;=\; (\mu_t - r_t + \lambda_t^{(1)} a_t^{(3)} \sigma(t,Y_t) x)^2 - 2\sigma^2(t,Y_t)(a_t^{(1)})^2 \left(\lambda_t^{(1)} [\ln(1+x) - x] + \lambda_t^{(2)} [\ln(1+y) - y]\right),$$

for fixed t in $[0,T]$. Let $\hat{x}$ be the solution of (5.2), it is unique since the function

$$x \longrightarrow 2(\lambda_t^{(1)})^2 \sigma^2(t,Y_t)(a_t^{(3)})^2 x + 2(a_t^{(1)})^2 \lambda_t^{(1)} \sigma^2(t,Y_t)\frac{x}{1+x} + 2\lambda_t^{(1)} \sigma(t,Y_t) a_t^{(3)}(\mu_t - r_t),$$

is strictly increasing from $]-1,\infty[$ to $\mathbb{R}$. Let $F_x^{'}$ and $F_y^{'}$ denote the first order partial derivatives of F. One can check that $(\hat{x}, 0)$ is the only point which satisfy $F_x^{'}(x,y) = F_y^{'}(x,y) = 0$. Moreover we have

$$(F_{xy}^{''}(\hat{x},0))^2 - F_{x^2}^{''}(\hat{x},0) F_{y^2}^{''}(\hat{x},0) < 0 \quad \text{and} \quad F_{x^2}^{''}(\hat{x},0) > 0.$$

Therefore F has a strict local minimum at $(\hat{x}, 0)$. This minimum is global since the limits of F on borders go to infinity. □

To obtain explicit formulas for the strategy computed in Proposition 4.3, we will separate the two cases : Brownian motion and Poisson process. In the two following subsections we compute explicitly the strategy for a European call option in the Brownian motion and the Poisson process cases respectively. The payoff of the option is then given by $f(S_T) = (S_T - K)^+$, where K denotes the Strike.

5.1 Brownian case

Assume that $a_t^{(1)} = a_t^{(2)} = 1$ and $a_t^{(3)} = a_t^{(4)} = 0$, so $(S_t)_{0\le t\le T}$ depends on Brownian information only. Under $\hat{Q}$, $(S_t)_{0\le t\le T}$ is given by

$$S_t = x \exp\left(\int_0^t \left(r_s - \frac{\sigma^2(s,Y_s)}{2}\right) ds + \int_0^t \sigma(s,Y_s) d\hat{W}_s^{(1)}\right),$$

with

$$Y_t = y + \int_0^t \left(\mu_s^Y + \sigma_s^{(1)} \frac{r_s - \mu_s}{\sigma(s, Y_s)} \right) ds + \int_0^t \sigma_s^{(1)} d\hat{W}_t^{(1)} + \int_0^t \sigma_s^{(2)} dW_s^{(2)}.$$

In the following proposition We compute the Malliavin derivative of the payoff $(S_T - K)^+$. We can replace the result in the formula (4.4), and obtain an explicit formula for the strategy.

Proposition 5.2 *We have*

$$\begin{aligned} D_t^{\hat{W}^{(1)}} (S_T - K)^+ &= 1_{\{S_T > K\}} S_T \Bigg(\sigma(t, Y_t) + \int_t^T \frac{\partial \sigma}{\partial y}(s, Y_s) D_s^{\hat{W}^{(1)}} Y_t d\hat{W}_s^{(1)} \\ &\quad - \int_t^T \sigma(s, Y_s) \frac{\partial \sigma}{\partial y}(s, Y_s) D_t^{\hat{W}^{(1)}} Y_s ds \Bigg) \end{aligned} \tag{5.3}$$

where

$$D_t^{\hat{W}^{(1)}} Y_s = \sigma_t^{(1)} \exp\left(- \int_t^s \sigma_u^{(1)} \frac{r_u - \mu_u}{\sigma^2(u, Y_u)} du \right) \quad s \in [t, T]. \tag{5.4}$$

Proof. By the chain role of $D_t^{\hat{W}^{(1)}}$ and thanks to Proposition 4.1 we obtain

$$\begin{aligned} & D_t^{\hat{W}^{(1)}} (S_T - K)^+ = \\ & \quad 1_{\{S_T > K\}} S_T \left(D_t^{\hat{W}^{(1)}} \int_0^T \left(r_s - \frac{\sigma^2(s, Y_s)}{2} \right) ds + D_t^{\hat{W}^{(1)}} \int_0^T \sigma(s, Y_s) d\hat{W}_s^{(1)} \right) \\ & = \quad 1_{\{S_T > K\}} S_T \left(- \int_t^T D_t^{\hat{W}^{(1)}} \frac{\sigma^2(s, Y_s)}{2} ds + \int_t^T D_t^{\hat{W}^{(1)}} \sigma(s, Y_s) d\hat{W}_s^{(1)} + \sigma(t, Y_t) \right), \end{aligned}$$

which gives (5.3). Concerning the other derivative, we have for $0 \leq t \leq s \leq T$

$$\begin{aligned} D_t^{\hat{W}^{(1)}} Y_s &= \int_t^s D_t^{\hat{W}^{(1)}} \left(\mu_u^Y + \sigma_u^{(1)} \frac{r_u - \mu_u}{\sigma(u, Y_u)} \right) du + \sigma_t^{(1)} \\ &= \sigma_t^{(1)} - \int_t^s \sigma_u^{(1)} \frac{r_u - \mu_u}{\sigma^2(u, Y_u)} D_t^{\hat{W}^{(1)}} Y_u du, \end{aligned}$$

So for t fixed in $[0, T]$, the Malliavin derivative of Y_s for $s \in [t, T]$: $(D_t^{\hat{W}^{(1)}} Y_s)_{s \in [t,T]}$, satisfies a stochastic differential equation, its solution is precisely (5.4). $\square$

5.2 The Poisson case

Similarly, like in the Brownian case, we aim to compute the quantity $D_t^{\hat{M}^{(1)}} (S_T - K)^+$ and replace the result in the expression of the strategy, to obtain an explicit formula for the Poisson case. Let us suppose that we work in the Poisson space with a 2-dimensional Poisson process. The underlying asset price $(S_t)_{0 \leq t \leq T}$ depends on Poisson process only. Hence we assume that $a_t^{(3)} = a_t^{(4)} = 1$ and $a_t^{(1)} = a_t^{(2)} = 0$. Under $\hat{Q}$, the dynamic of $(S_t)_{0 \leq t \leq T}$ is given by

$$S_t = x \exp\left(\int_0^t \left(\mu_s + \frac{r_s - \mu_s}{\sigma(s, Y_s)} \ln(1 + \sigma(s, Y_s)) \right) ds + \int_0^t \ln(1 + \sigma(s, Y_s)) d\hat{M}_s^{(1)} \right),$$

for $t \in [0,T]$. The process $(Y_t)_{t\in[0,T]}$ under $\hat{Q}$, have the representation

$$Y_t = y + \int_0^t \left(\mu_s^Y + \sigma_s^{(1)} \frac{r_s - \mu_s}{\sigma(s,Y_s)}\right) ds + \int_0^t \sigma_s^{(1)} d\hat{M}_t^{(1)} + \int_0^t \sigma_s^{(2)} dM_s^{(2)}.$$

Proposition 5.3

$$\begin{aligned} & D_t^{\hat{M}^{(1)}}(S_T - K)^+ = \\ & -(S_T - K)^+ + \Bigg(\exp\left\{\int_t^T \left[\mu_s + \frac{r_s - \mu_s}{\sigma(s, Y_s + \sigma_t^{(1)})} \ln(1 + \sigma(s, Y_s + \sigma_t^{(1)}))\right] ds \right. \\ & \left. + \int_t^T \ln(1 + \sigma(s, Y_s + \sigma_t^{(1)})) d\hat{M}_s^{(1)} \right\} \times S_t(1 + \sigma(t, Y_t + \sigma_t^{(1)})) - K\Bigg)^+ \end{aligned}$$

Proof. Using the probabilistic interpretation of $D_t^{\hat{M}^{(1)}}$ given below, we obtain

$$D_t^{\hat{M}^{(1)}}(S_T - K)^+ = (S_T(\omega + 1_{[t,T]}) - K)^+ - (S_T(\omega) - K)^+.$$

But

$$\begin{aligned} S_T(\omega + 1_{[t,T]}) = {} & x \exp\left(\int_0^t \left[\mu_s + \frac{r_s - \mu_s}{\sigma(s, Y_s(\omega + 1_{[t,T]}))} \ln(1 + \sigma(s, Y_s(\omega + 1_{[t,T]})))\right] ds \right. \\ & \left. + \int_0^t \ln(1 + \sigma(s, Y_s(\omega + 1_{[t,T]}))) d\hat{M}_s^{(1)}\right) \\ & \times \exp\left(\int_t^T \left[\mu_s + \frac{r_s - \mu_s}{\sigma(s, Y_s(\omega + 1_{[t,T]}))} \ln(1 + \sigma(s, Y_s(\omega + 1_{[t,T]})))\right] ds \right. \\ & \left. + \int_t^T \ln(1 + \sigma(s, Y_s(\omega + 1_{[t,T]}))) d\hat{M}_s^{(1)}\right) \times (1 + \sigma(t, Y_t(\omega + 1_{[t,T]}))), \end{aligned}$$

and

$$\begin{aligned} Y_t(\omega + 1_{[t,T]}) &= Y_t + \sigma_t^{(1)}, \quad t \in [0,T], \\ Y_s(\omega + 1_{[t,T]}) &= \begin{cases} Y_s & \text{if} \quad s \in [0,t[, \\ Y_s + \sigma_t^{(1)} & \text{if} \quad s \in [t,T]. \end{cases} \end{aligned}$$

The proof is established. □

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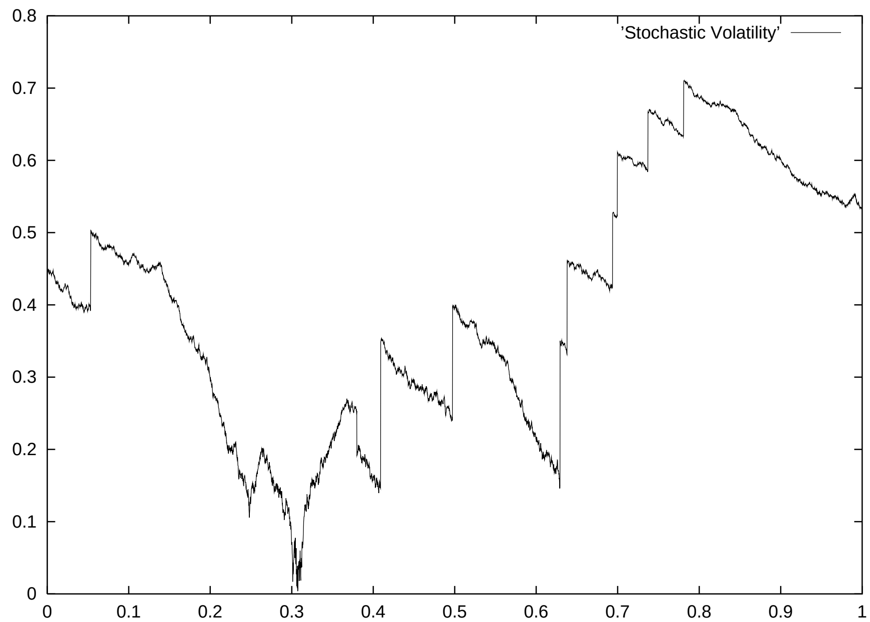

Figure 1: A sample trajectory of the volatility $\sigma(t, Y_t)$.

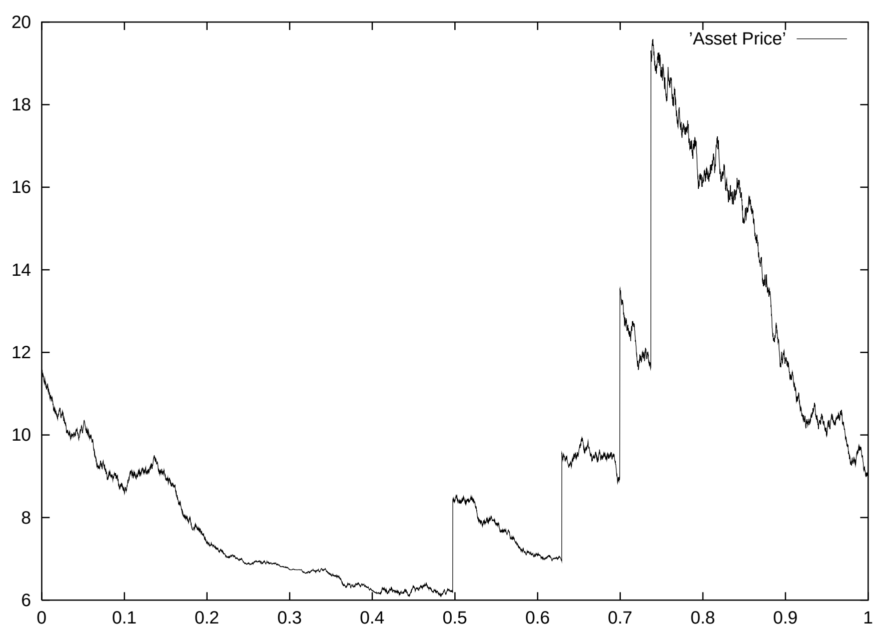

Figure 2: A sample trajectory of the price S_t.